STABILITY ANALYSIS OF IMPULSIVE NEURAL NETWORKS WITH SUPREMUM

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***Abstract**: In this paper, we study the problem of global asymptotic stability of the equilibrium of a class of neural networks with supremum and fixed moments of impulsive effect. We establish stability criteria by employing Lyapunov functions and Razumikhin technique.*

***Keywords**: Neural networks, Impulses, Global asymptotic stability, Lyapunov method, Supremum.*

1. INTRODUCTION

Since cellular neural networks (CNNs) were introduced by Chua and Yang in 1988 [3], they have been widely studied both in theory and applications [4,6,7]. Although electronic circuits of CNNs can be fabricated into chips by very large scale integration technology, the finite switching speed of amplifiers and communication time will introduce the time delays in the interaction among the cells. Moreover, to process moving images, one must introduce the time delays in the signals transmitted among the cells. These lead to the model of CNNs with delay (DCNNs). They have found applications in different areas such as classification of patterns and reconstruction of moving images.

Motivated by the above consideration, Gopalsamy and Leung [5] considered the following scalar autonomous delay equation with dynamical thresholds:

$$\dot{x}(t) = -x(t) + a\tanh(x(t) - bx(t-\tau) - c), \quad t \ge 0$$

Where $x : R_+ \to R$, a is a positive constant, b, c and τ are nonnegative constants. By using Lyapunov functions, Gopalsamy and Leung established a sufficient condition for global asymptotic stability of the equilibrium $x^* = 0$ for the case $c = 0$.

For the case $c \ne 0$ some stability criteria are investigated in [11] for the equilibrium of the following more general model:

$$\dot{x}(t) = -x(t) + a\,f\,(x(t) - bx(t-\tau) - c), \qquad (1)$$

$t \ge 0$, $f : R \to R$

On the other hand, the state of electronic networks is often subject to instantaneous perturbations and experience abrupt changes at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise, that exhibit impulsive effects [1,2,6,8,9]. Impulses can make unstable systems stable so they have been widely used in many fields such as physics, chemistry, biology, population dynamics, and industrial robotics.

In the mathematical simulation in various important applicable branches one has to analyses the influence of both the maximum of the function investigated and its impulsive changes. An adequate mathematical apparatus for simulation of such processes are the impulsive differential equations with supremums.

In this paper, we study the global asymptotic stability of the following impulsive generalization of the model (1) with supremum

$$\begin{cases} \dot{x}(t) = -x(t) + a\,f\,(x(t) \\ \quad - b \sup\limits_{s\in[t-\tau,t]} x(s) - c), \quad t \ne t_k, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, ..., \end{cases} \qquad (2)$$

Where: $t \ge 0$, $\Delta x(t_k) = x(t_k + 0) - x(t_k)$, $I_k : R \to R$, $k = 1, 2, ...$,

$0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots, \ \lim_{k\to\infty} t_k = \infty$.

The numbers $x(t_k)$ and $x(t_k + 0)$ are respectively, the states of the network before and after the impulsive perturbation at the moment t_k, and the functions $I_k(x)$ characterize the magnitude of the impulse effect at the moments t_k.

By using of piecewise continuous Lyapunov functions and the Razumikhin technique [9,10] we establish criteria for global asymptotic stability.

2. STATEMENT OF THE PROBLEM. PRELIMINARIES

Let $R_+ = [0,\infty)$ and $J \subseteq R$. Define the following class of functions:

$PCB[J,R] = \{\sigma \in PC[J,R]: \ \sigma(t)$ is bounded on $J\}$.

Let $\varphi \in PCB[[-\tau,0],R]$. Denote by $x(t) = x(t;0,\varphi), \ x \in R$ the solution of (2), satisfying the initial conditions:

$$\begin{cases} x(t;0,\varphi) = \varphi(t), \quad t \in [-\tau,0], \\ x(0^+;\,0,\varphi) = \varphi(0) \end{cases} \qquad (3)$$

Let $|\varphi|_\tau = \sup_{s\in[-\tau,0]} |\varphi(s)|$ be the norm of the function $\varphi \in PCB[[-\tau,0],R]$.

Introduce the following conditions:

H1. There exists a constant $L > 0$ such that $|f(u) - f(v)| \le L|u - v|$ for all $u, v \in R$.

H2. There exists a constant $M > 0$ such that for all $u \in R$, $|f(u)| \le M < \infty$.

H3. $a > 0, \ b \ge 0, \ a(1-b) < 1$.

H4. For any $k = 1,2,\ldots,$ the functions I_k are continuous in R.

H5. $0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$ and $t_k \to \infty$ for $k \to \infty$.

Let $y(t) = x(t) - b\sup_{s\in[t-\tau,t]} x(s) - c$. We transform (2) into the form

$$\begin{cases} \dot{y}(t) = -y(t) - c + a f(y(t)) \\ \quad - abf(\sup_{s\in[t-\tau,t]} y(s)), \quad t \ne t_k, \\ \Delta y(t_k) = J_k(y(t_k)), \quad k = 1,2,\ldots, \end{cases} \qquad (4)$$

Where:

$$J_k(y(t_k)) = I_k(y(t_k) + b\sup_{s\in[t-\tau,t]} x(t_k) + c) - I_k(b\sup_{s\in[t-\tau,t]} x(t_k) + c, \ k = 1,2,\ldots.$$

We will use the following lemma.

Lemma 1. Let the conditions H1-H5 hold. Then:

(a) there exists a unique equilibrium point x^* of (2), defined for $t \in [0,\infty)$.

(b) $\lim_{t\to\infty} x(t) = x^*$ as $\lim_{t\to\infty} y(t) = y^*$,

where y^* is the equilibrium of (4).

Proof. Under the hypotheses H1-H3, the equation without impulses

$$\dot{x}(t) = -x(t) + a f(x(t) - b\sup_{s\in[t-\tau,t]} x(s) - c), \quad t \ge 0$$

has [9,11] a unique equilibrium x^* on the interval $[0,\infty)$. That means that the solution x^* of (2) is defined on each of the intervals $(t_{k-1}, t_k], \ k = 1,2,\ldots$. From the conditions H4 and H5 we conclude that it is continuable for $t \ge 0$. The proof of Lemma 1 follows from the fact that if y^* denotes an equilibrium of the equation (4), then:

$$J_k(y^*) = 0, \quad k = 1,2,\ldots$$

Remark 1. The problems of existence, uniqueness, and continuability of the solutions of impulsive functional differential equations has been investigated in the monograph [9].

Definition 1. The equilibrium x^* of the equation (2) is said to be:

a) *globally stable*, if

$(\forall \varepsilon > 0) \ (\exists \delta = \delta(\varepsilon) > 0)$

$(\forall \varphi \in PCB[[-\tau,0],R]: \ |\varphi - x^*|_\tau < \delta)$

$(\forall t \ge 0): \ |x(t;0,\varphi) - x^*| < \varepsilon$;

b) *globally asymptotically stable*, if it is globally stable and

$\lim_{t\to\infty} x(t;0,\varphi) = x^*$.

3. MAIN RESULTS

In this section we will prove sufficient conditions for the global asymptotic stability of the equilibrium x^* of (2). We will use piecewise continuous Lyapunov functions.

Introduce the class V_0 of all functions $V:[0,\infty)\times R \to R_+$, which are continuous in $(t_{k-1},t_k)\times R, k=1,2,...$ and locally Lipschitz continuous on R, there exist the finite limits:

$$\lim_{\substack{t\to t_k\\ t<t_k}} V(t,x) = V(t_k-0,x)$$

$$\lim_{\substack{t\to t_k\\ t>t_k}} V(t,x) = V(t_k+0,x)$$

and $V(t_k,x)=V(t_k-0,x)$ for all $k=1,2,...$

For $t\neq t_k, k=1,2,...$ and $V\in V_0$ we define

$$D^+V(t,x(t)) =$$
$$= \limsup_{h\to 0^+}\frac{1}{h}[V(t+h,x(t+h))-V(t,x(t))].$$

Set $u(t)=y(t)-y^*$ and consider the following equation

$$\begin{cases} \dot u(t) = -u(t) \\ +a[f(u(t)+y^*)-f(y^*)] \\ -ab[f(\sup\limits_{s\in[t-\tau,t]} u(s)+y^*) \\ -f(y^*)], \quad t\neq t_k, \\ \Delta u(t_k) = P_k(u(t_k)), \end{cases} \tag{5}$$

Where: $t>0$,

$P_k(u) = J_k(u+y^*) - J_k(y^*) = J_k(u+y^*)$,

$k=1,2,...$.

Theorem 1. Assume that:

1. Conditions H1-H5 hold.
2. There exists a constant d such that: $0<d\le 1-La(1+b)$.
3. The functions P_k are such that: $P_k(u(t_k)) = -\sigma_k u(t_k)$, $\quad 0<\sigma_k<2$, $k=1,2,...$

Then the equilibrium x^* of the equation (2) is globally asymptotically stable.

Proof. We define a Lyapunov function:

$$V(t,u)=\frac{1}{2}u^2.$$

Then for $t\ge 0$ and $t=t_k$, from the condition 3 of Theorem 1, we obtain:

$$V(t+0,u(t)+P_k(u(t))) = \frac{1}{2}(u(t)+P_k(u(t)))^2 =$$

$$= \frac{1}{2}(1-\sigma_k)^2u^2(t) < V(t,u(t)), \tag{6}$$

$k=1,2,...$.

Let $t\ge 0$ and $t\neq t_k$. Then for the upper right-hand derivative $D^+V(t,u(t))$ of the function $V(t,u(t))$ with respect to equation (5) we get:

$$D^+V(t,u(t)) = u(t).\dot u(t) =$$
$$= u(t)(-u(t)+a[f(u(t)+y^*)-f(y^*)]-$$
$$-ab[f(\sup_{s\in[t-\tau,t]} u(s)+y^*)-f(y^*)]) =$$
$$= -u^2(t)+a\,u(t)[f(u(t)+y^*)-f(y^*)]-$$
$$-abu(t)[f(\sup_{s\in[t-\tau,t]} u(s)+y^*)-f(y^*)]$$

Since for the function f assumption H1 is true, then for $t\ge 0$ and $t\neq t_k$ we have:

$$f(u(t)+y^*)-f(y^*) = u(t)f'(\xi_1(t)),$$
$$f(\sup_{s\in[t-\tau,t]} u(s)+y^*)-f(y^*) =$$
$$= \sup_{s\in[t-\tau,t]} u(s)f'(\xi_2(t))$$

where $\xi_1(t)$ lies between y^* and $u(t)+y^*$, and $\xi_2(t)$ lies between y^* and $u(t-\tau(t))$.

Then

$$D^+V(t,u(t)) \le -u^2(t),$$
$$+aLu^2(t)+abLu(t)\sup_{s\in[t-\tau,t]} u(s)$$

for $t\ge 0$ and $t\neq t_k$.

From the last estimate for any solution $u(t)$ of (5), that satisfies the Razumikhin condition [9,10]:

$V(t,u(s)) \le V(t,u(t))$, $\; t-\tau\le s\le t$,

$t\ge 0$, $t\neq t_k$,

we have:

$$D^+V(t,u(t)) \le$$
$$\le (-1+aL(1+b))V(t,u(t)) = -dV(t,u(t)),$$

where, by condition 2 of Theorem1 $d>0$.

From the above inequality and (6), we have

$$V(t,u(t)) \le e^{-dt}V(0,u(0)),\ t\ge 0.$$

which means that the zero solution of the equation (5) is globally asymptotically stable.

Therefore, the equilibrium x^* of the equation (2) is globally asymptotically stable.

4. CONCLUSIONS

In this paper, by using a suitable piecewise continuous Lyapunov function and the Razumikhin technique, the sufficient condition for global asymptotic stability of the equilibrium point of a neural network model with supremum and nonlinear impulsive operators is obtained. Since delays and impulses can affect the dynamical behaviors of the system, it is necessary to investigate both delay and impulsive effects on the stabilization of neural networks. These play an important role in the design and applications of asymptotically stable neural networks with delays. The technique can be extended to study other impulsive delayed systems.

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